# Measurement of Length

## OBJECTIVES

❑ Demonstrate the specific knowledge gained by repeated measurements of the length and width of a table.

❑ Apply the statistical concepts of mean, standard deviation from the mean, and standard error to these measurements.

❑ Demonstrate propagation of errors by determining the uncertainty in the area calculated from the measured length and width.

## EQUIPMENT LIST

- 2-meter stick
- Laboratory table

## THEORY

In this laboratory it is assumed that the uncertainty in the measurement of the length and width of the table is due to random errors. If this assumption is valid, then the mean of a series of repeated measurements represents the most probable value for the length or width.

Consider the general case in which $n$ measurements of the length and width of the table are made. We will make 10 measurements, so $n = 10$ for this case, but we will develop equations for the case in which $n$ can be any chosen value. If $L_i$ and $W_i$ stand for the individual measurements of the length and width, and $\bar{L}$ and $\bar{W}$ stand for the **mean** of those measurements, the equations relating them are

$$\bar{L} = \left(\frac{1}{n}\right)\sum_{i}^{n} L_i \qquad \bar{W} = \left(\frac{1}{n}\right)\sum_{i}^{n} W_i \qquad \text{(Eq. 1)}$$

We get information about the precision of the measurement from the variations of the individual measurements using the statistical concept of the standard deviation. The values of the **standard deviation** from the mean for the length and width of the table, $\sigma_{n-1}^{L}$ and $\sigma_{n-1}^{W}$, are given by the equations:

$$\sigma_{n-1}^{L} = \sqrt{\frac{1}{n-1}\sum_{1}^{n}(L_i - \bar{L})^2} \qquad \sigma_{n-1}^{W} = \sqrt{\frac{1}{n-1}\sum_{1}^{n}(W_i - \bar{W})^2} \qquad \text{(Eq. 2)}$$

If errors are only random, it should be true that approximately 68.3% of the measurements of length should fall in the range $\overline{L} \pm \sigma_{n-1}^{L}$, and that approximately 68.3% of the measurements of width should fall within the range $\overline{W} \pm \sigma_{n-1}^{W}$. Furthermore, 95.5% of the measurements of both length and width should fall within 2 $\sigma_{n-1}$ of the mean, and 99.73% should fall within 3 $\sigma_{n-1}$ of the mean.

The precision of the mean for $\overline{L}$ and $\overline{W}$ is given by quantities called the **standard error**, $\alpha_L$ and $\alpha_W$. These quantities are defined by the following equations:

$$\alpha_L = \frac{\sigma_{n-1}^{L}}{\sqrt{n}} \qquad \alpha_W = \frac{\sigma_{n-1}^{W}}{\sqrt{n}} \qquad\qquad \text{(Eq. 3)}$$

The meaning of $\alpha_L$ and $\alpha_W$ is that, if the errors are only random, there is a 68.3% chance that the true value of the length lies within the range $\overline{L} \pm \alpha_L$ and the true value of the width lies within the range $\overline{W} \pm \alpha_W$.

An important problem in experimental physics is to determine the uncertainty in some quantity that is derived by calculations from other directly measured quantities. For this experiment, consider the area $A$ of the table as calculated from the measured values of the length and width $\overline{L}$ and $\overline{W}$ by the following:

$$A = \overline{L} \times \overline{W} \qquad\qquad \text{(Eq. 4)}$$

For the case of an area that is the product of two measured quantities, the uncertainty in the area is related to the uncertainty of the length and width by:

$$\alpha_A = \sqrt{(\overline{L})^2 (\alpha_W)^2 + (\overline{W})^2 (\alpha_L)^2} \qquad\qquad \text{(Eq. 5)}$$

## EXPERIMENTAL PROCEDURE

1. Place the 2-meter stick along the length of the table near the middle of the width and parallel to one edge of the length. Do not attempt to line up either edge of the table with one end of the meter stick or with any certain mark on the meter stick.

2. Let $X$ stand for the coordinate position in the length direction. Read the scale on the 2-meter stick that is aligned with one end of the table and record that measurement in Data and Calculations Table 1 as $X_1$. Read the scale that is aligned at the other end of the table and record that measurement in Data and Calculations Table 1 as $X_2$. A 3×5 note card held next to the edge of the table may help to determine where the 2-meter stick is aligned with the table for each measurement. Note that the stick has 1 millimeter as the smallest marked scale division. *Therefore, each coordinate should be estimated to the nearest 0.1 millimeter (nearest 0.0001 m).*

3. Repeat Steps 1 and 2 nine more times for a total of 10 measurements of the length of the table. For each measurement place the 2-meter stick on the table with no attempt to align either end of the stick or any particular mark on the stick with either end of the table. Make the measurements at 10 different places along the width of the table so that any variation in the length of the table is included in the measurements.

4. Perform Steps 1 through 3 for 10 measurements of the width of the table. Let the coordinate for the width be given by $Y$ and record the 10 values of $Y_1$ and $Y_2$ in Data and Calculations Table 2. Again place the stick along the different lines each time, but make no attempt to align any particular mark on the stick with either edge of the table.

# CALCULATIONS

1. After all measurements are completed, perform the subtractions of the coordinate positions to determine the 10 values of the length $L_i$, and the 10 values of width $W_i$. Record the 10 values of $L_i$ and $W_i$ in the appropriate table.

2. Use Equations 1 to calculate the mean length $\overline{L}$ and the mean width $\overline{W}$ and record their values in the appropriate table. Keep five decimal places in these results. For example, typical values might be $\overline{L} = 1.37157\,\text{m}$ and $\overline{W} = 0.76384\,\text{m}$.

3. For each measurement of length and width, calculate the values of $L_i - \overline{L}$ and $W_i - \overline{W}$ and record them in the appropriate table. Then for each value of the length and width, calculate and record the values of $(L_i - \overline{L})^2$ and $(W_i - \overline{W})^2$ in the appropriate table.

4. Perform the summations of the values of $(L_i - \overline{L})^2$ and the summations of the values of $(W_i - \overline{W})^2$ and record them in the appropriate box in the tables.

5. Use the values of the summations of $(L_i - \overline{L})^2$ and of $(W_i - \overline{W})^2$ in Equations 2 to calculate the values of $\sigma^L_{n-1}$ and $\sigma^W_{n-1}$ and record them in the appropriate table.

6. Calculate $\overline{L} - \sigma^L_{n-1}$, $\overline{L} + \sigma^L_{n-1}$, $\overline{W} - \sigma^W_{n-1}$, and $\overline{W} + \sigma^W_{n-1}$ and record the values in the appropriate table.

7. Use the values of $\sigma^L_{n-1}$ and $\sigma^W_{n-1}$ in Equations 3 to calculate the values of $\alpha_L$ and $\alpha_W$ and record them in the appropriate table.

8. Use the values of $\overline{L}$ and $\overline{W}$ in Equation 4 to calculate the value of $A$, the area of the table, and record it in the appropriate table. Use Equation 5 to calculate the value of $\alpha_A$ and record it in the appropriate table.

# LABORATORY 1    *Measurement of Length*

## PRE-LABORATORY ASSIGNMENT

1. State the number of significant figures in each of the following numbers and explain your answer.
   - (a) 37.60_____
   - (b) 0.0130_____
   - (c) 13000_____
   - (d) 1.3400_____

2. Perform the indicated operations to the correct number of significant figures using the rules for significant figures.

   (a)   $\begin{array}{r} 37.60 \\ \times\ 1.23 \\ \hline \end{array}$     (b)   $6.7\overline{\smash{)}8.975}$     (c)   $\begin{array}{r} 3.765 \\ +\ 1.2 \\ +37.21 \\ \hline \end{array}$

3–6. Three students named Abe, Barb, and Cal make measurements (in m) of the length of a table using a meter stick. Each student's measurements are tabulated in the table below along with the mean, the standard deviation from the mean, and the standard error of the measurements.

| Student | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $\overline{L}$ | $\sigma_{n-1}$ | $\alpha$ |
|---------|-------|-------|-------|-------|--------|----------|----------|
| Abe | 1.4717 | 1.4711 | 1.4722 | 1.4715 | 1.4716 | 0.00046 | 0.0002 |
| Barb | 1.4753 | 1.4759 | 1.4756 | 1.4749 | 1.4754 | 0.00043 | 0.0002 |
| Cal | 1.4719 | 1.4723 | 1.4727 | 1.4705 | 1.4719 | 0.00096 | 0.0005 |

Note that in each case only one significant figure is kept in the standard error $\alpha$, and this determines the number of significant figures in the mean. The actual length of the table is determined by very sophisticated laser measurement techniques to be 1.4715 m.

3. State how one determines the accuracy of a measurement. Apply your idea to the measurements of the three students above and state which of the students has the most accurate measurement. Why is that your conclusion?

4. Apply Equations 1, 2, and 3 to calculate the mean, standard deviation, and standard error for Abe's measurements of length. Confirm that your calculated values are the same as those in the table. Show your calculations explicitly.

5. State the characteristics of data that indicate a systematic error. Do any of the three students have data that suggest the possibility of a systematic error? If so, state which student it is, and state how the data indicate your conclusion.

6. Which student has the best measurement considering both accuracy and precision? State clearly what the characteristics are of the student's data on which your answer is based.

**LABORATORY 1**   *Measurement of Length*

# LABORATORY REPORT

*Data and Calculations Table 1 (nearest 0.0001 m, which is 0.1 mm)*

| Trial | $X_1$ (m) | $X_2$ (m) | $L_i = X_2 - X_1$ (m) | $L_i - \overline{L}$ (m) | $(L_i - \overline{L})^2$ (m$^2$) |
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|  |  |  |  | $\sum_{1}^{n} (L_i - \overline{L})^2 =$ |  |

$\overline{L} =$ _____   $\sigma_{n-1}^{L} =$ _____   $\overline{L} - \sigma_{n-1}^{L} =$ _____   $\overline{L} + \sigma_{n-1}^{L} =$ _____   $\alpha_L =$ _____

*Data and Calculations Table 2 (nearest 0.0001 m, which is 0.1 mm)*

| Trial | $Y_1$ (m) | $Y_2$ (m) | $W_i = Y_2 - Y_1$ (m) | $W_i - \overline{W}$ (m) | $(W_i - \overline{W})^2$ (m$^2$) |
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| | | | | $\sum_{1}^{n} (W_i - \overline{W})^2 =$ | |

$\overline{W} = \underline{\hspace{2cm}}$   $\sigma_{n-1}^{W} = \underline{\hspace{2cm}}$   $\overline{W} - \sigma_{n-1}^{W} = \underline{\hspace{2cm}}$   $\overline{W} + \sigma_{n-1}^{W} = \underline{\hspace{2cm}}$   $\alpha_W = \underline{\hspace{2cm}}$

$A = \overline{L} \times \overline{W} = \underline{\hspace{5cm}}$   $\sigma_A = \underline{\hspace{5cm}}$

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# SAMPLE CALCULATIONS

**1.** $L_1 = X_2^1 - X_1^1 =$

**2.** $W_1 = Y_2^1 - Y_1^1 =$

**3.** $\overline{L} = \dfrac{1}{10} \sum_{1}^{10} L_i =$

**4.** $\overline{W} = \dfrac{1}{10} \sum_{1}^{10} W_i =$

**5.** $L_1 - \overline{L} =$

**6.** $(L_1 - \overline{L})^2 =$

**7.** $W_1 - \overline{W} =$

**8.** $(W_1 - \overline{W})^2 =$

**9.** $\sigma_{n-1}^{L} = \sqrt{\dfrac{1}{n-1} \sum_{1}^{n} (L_i - \overline{L})^2}$

**10.** $\overline{L} - \sigma_{n-1}^{L} =$

**11.** $\overline{L} + \sigma_{n-1}^{L} =$

**12.** $\sigma_{n-1}^W = \sqrt{\dfrac{1}{n-1} \sum_{1}^{n} (W_i - \overline{W})^2}$

**13.** $\overline{W} - \sigma_{n-1}^W =$

**14.** $\overline{W} + \sigma_{n-1}^W =$

**15.** $A = \overline{L} \times \overline{W} =$

**16.** $\sigma_A =$

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## QUESTIONS

1. According to statistical theory, 68% of your measurements of the length of the table should fall in the range from $\overline{L} - \sigma_{n-1}^L$ to $\overline{L} + \sigma_{n-1}^L$. About 7 of your 10 measurements should fall in this range. What is the range of these values for your data? From _____m to _____m. How many of your 10 measurements of the length of the table fall in this range? _____? State clearly the extent to which your data for the length agree with the theory. What is your evidence for your statement?

2. Answer the same question for the width. Range of $\overline{W} - \sigma_{n-1}^W$ to $\overline{W} + \sigma_{n-1}^W$ is from _____m to _____m. The number of measurements that fall in that range is _____. Do your data for the width of the table agree with the theory reasonably well? State your evidence for your opinion.

3. According to statistical theory, if any measurement of a given quantity has a deviation greater than $3\sigma_{n-1}$ from the mean of that quantity, it is very unlikely that it is statistical variation, but rather is more likely to be a mistake. Calculate the value of $3\sigma_{n-1}^L$. Do any of your measurements of length have a deviation from the mean greater than that value? If so, calculate how many times larger than $\sigma_{n-1}^L$ it is. Do any of your measurements of the length appear to be a mistake, and, if so, which ones?

4. For the width measurements calculate $3\sigma_{n-1}^W$. Do any of your measurements of width have a deviation from the mean greater than that value? If so, calculate how many times larger than $\sigma_{n-1}^W$ it is. Do any of your measurements of width appear to be a mistake, and, if so, which ones?

**5.** If possible, state the accuracy of your measurements of the length and width and give your reasoning. If this cannot be done, state why it is not possible. If possible, state the precision of your measurement of the length and width and give your reasoning. If this cannot be done, state why it is not possible.